prelims review sheet:
fifty pages of formulas
that you should have memorized

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1 Classical Mechanics

1.1 Lagrangians Overview

Assuming Newton's second, we derive the euler lagrange equations

$$m\ddot{\mathbf{r}}_\alpha = -\nabla V\Big|_{\mathbf{r}_\alpha} + \mathbf{F}_\alpha^{\text{n-c}} \implies \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \sum \frac{\partial \mathbf{r}_\alpha}{\partial q} \cdot \mathbf{F}_\alpha^{\text{n-c}}$$

where we absorb the conservative forces into L. Some nonconservative forces can also be written as a velocity-dependent potential; an example is the Rayleigh dissipation function

$$\mathcal{F} = \frac{1}{2}\sum (k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2)$$

for which the force is given by

$$\frac{\partial \mathbf{r}_\alpha}{\partial q} \cdot \mathbf{F}_\alpha^{\text{n-c}} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}$$

if there is a dissipation function $\mathcal{F}$, we have

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -2\mathcal{F} - \frac{\partial L}{\partial t}$$

as the rate of energy loss.

1.1.1 Constraints and Lagrange Multipliers

Fun vocabulary:

- A *holonomic* constraint is one that can be written $f(\mathbf{r}_1, \mathbf{r}_2, \ldots; t) = 0$
- A *rheonomous* constraint is one that contains a dependence on time. Otherwise, the constraint is called *scleronomous*
- A *semiholonomic* constraint is one that can be written in the form

$$f(q, \dot{q}) = 0 \quad \text{or, often,} \quad a_i \, \mathrm{d}q_i + a_t \, \mathrm{d}t = 0$$

We include these constraints using Lagrange multipliers:

1. Since the constraint equations f_α are identically zero, we can say

$$\delta \int (L - \lambda^\alpha(t) f_\alpha) \, \mathrm{d}t = 0$$

2. Solving the variation gives the nonhomogeneous Euler-Lagrange equations for forces of constraint

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \sum \lambda^\alpha \left[\frac{\partial f_\alpha}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f_\alpha}{\partial \dot{q}} \right] - \frac{\mathrm{d}\lambda^\alpha}{\mathrm{d}t}\frac{\partial f_\alpha}{\partial \dot{q}}$$

Note that this right hand side simplifies a lot if the constraint is holonomic

1.1.2 Conserved quantities

cyclic coords (lagrangian does not explictly depend on them) immediately give conserved quantities, and typically energy is conserved as well.

Noethers theorem says

$$\text{if} \qquad q_i \to q_i + \delta q_i \implies \delta L = \frac{\mathrm{d}f}{\mathrm{d}t} \quad \text{for some} \quad f(q, t)$$

$$\text{then} \qquad Q = p_i \delta q^i - f \text{ is constant when the EoM is followed}$$

1.2 Hamiltonians

$$H(p, q) = p_i \dot{q}^i - L$$

with equations of motion

$$\frac{\partial H}{\partial q} = -\dot{p} \quad \text{and} \quad \frac{\partial H}{\partial p} = \dot{q}$$

1.3 Oscillations

1.3.1 One dimension: driving and damping

Consider the EoM

$$m\ddot{x} = -kx - \gamma\dot{x} + F_D(t)$$

define the undamped frequency and the quality factor

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad Q = \frac{\sqrt{mk}}{\gamma} \implies \ddot{x} + \frac{\omega_0}{Q}\dot{x} + \omega_0^2 x = \frac{1}{m}F_D$$

if we take $x \sim e^{rt}$ and solve the homogeneous, we get the characteristic equation

$$r^2 + \frac{\omega_0}{Q}r + \omega_0^2 = 0 \implies r_\pm = \frac{\omega_0}{2Q}\left[-1 \pm \sqrt{1-4Q^2}\right]$$

which has three cases:

- $Q > 1/2$ "underdamped" — decaying packet times an oscillatory solution

  $$x(t) = e^{-\omega_0 t/2Q}\cos\left(\omega_0\sqrt{1-1/4Q^2} + \phi\right)$$

- $Q = 1/2$ "critically damped"

  $$x(t) = [x_0 + (v_0 + x_0\omega_0)t]e^{-\omega_0 t}$$

  you can shove it hard enough to overshoot the equilibrium point but it'll approach it exponentially from there

- $Q > 1/2$ "overdamped" — real and distinct roots give a lincomb

  $$x(t) = Ae^{r_+ t} + Be^{r_- t}$$

to get a particular solution in the case of a driving force, generally nice to solve in the complex plane. $x = \mathrm{Re}[z(t)]$ for $z(t) = Ae^{-i\omega t}$ where ω is the driving frequency and A is in general complex. We can pull this out into a real amplitude and a phase lag. For a periodic driving force, we get resonance: $|A|$ is big when $\omega \sim \omega_0$ and Q large.

$$|A| \text{ maximized when } \omega = \omega_0\sqrt{1-1/2Q^2}$$

in $A-\omega$ space this gives a lorentzian with width $\Gamma = \omega/Q$ that looks like

$$A(\omega) \sim \frac{\Gamma^2/4}{(\omega-\omega_0)^2 + \Gamma^2/4}E_{\max}$$

1.3.2 Higher dimensions

If we have a Lagrangian of the form

$$L = \frac{1}{2}T_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}V_{ij}q_iq_j + O(q^3) = \frac{1}{2}\dot{\mathbf{q}}^T\mathbb{T}\dot{\mathbf{q}} - \frac{1}{2}\mathbf{q}^T\mathbb{V}\mathbf{q}$$

then the equations of motion read

$$T_{ij}\ddot{q}_j + V_{ij}q_j = 0 \implies q_j = v_j e^{i\omega t} \implies (-\omega^2\mathbb{T} + \mathbb{V})\mathbf{v} = 0$$

so we find valid ω^2s by $\det(-\omega^2\mathbb{T} - \mathbb{V}) = 0$, and the sign determines the time dependence of q:

- if $\omega_\alpha^2 > 0$, $\mathbf{q} = \mathbf{v}_\alpha[A_\alpha\cos(\omega t) + B_\alpha\sin(\omega t)]$
- if $\omega_\alpha^2 = 0$, $\mathbf{q} = \mathbf{v}_\alpha[A_\alpha t + B_\alpha]$
- if $\omega_\alpha^2 < 0$, $\mathbf{q} = \mathbf{v}_\alpha\left[A_\alpha e^{|\omega_\alpha|t} + B_\alpha e^{-|\omega_\alpha|t}\right]$

and the different $\mathbf{v}_\alpha$ are $\mathbb{T}$-orthornomal, meaning

$$\mathbf{v}_\alpha^T\mathbb{T}\mathbf{v}_\beta = \delta_{\alpha\beta}$$

so in a 2D system knowing one determines the other

1.4 The Central Force Problem

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\phi})^2 - V(r)$$

can transform to the hammy or to the routhian ($R = p_\phi\dot{\phi} - L$, but we leave r as in the lagrangian formalism)

$$H = \frac{p_r^2}{2m} + \underbrace{\frac{p_\phi^2}{2mr^2} + V(r)}_{V_{\text{eff}}}$$

we usually care about $r(\phi)$, so we convert the time derivatives to phi derivatives:

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\dot{\phi} = r'\frac{p_\phi}{mr^2} \implies E = \frac{p_\phi^2}{2m}\frac{r'^2}{r^4} + V_{\text{eff}}$$

note it's also often nice to swap variables to $u = 1/r$

1.4.1 The Kepler problem

$$\begin{aligned} E &= \frac{p_\phi^2}{2m}\frac{r'^2}{r^4} + \frac{p_\phi^2}{2mr^2} - \frac{k}{r} \\ &= \frac{p_\phi^2}{2m}u'^2 + \frac{p_\phi^2}{2m}u^2 - ku \\ &= \frac{p_\phi^2}{2m}\left[u'^2 + \left(u - \frac{km}{p_\phi^2}\right)^2\right] - \frac{mk^2}{2p_\phi^2} \end{aligned}$$

which is an offset SHO! customary names:

$$u_0 = \frac{km}{p_\phi^2} = \frac{1}{p} \qquad \varepsilon = Ap \implies r(\phi) = \frac{p}{1 + \varepsilon\cos(\phi - \phi_0)}$$

$$E = \frac{p_\phi^2}{2m}A^2 - \frac{mk^2}{2p_\phi^2} = \frac{k}{2p}(\varepsilon^2 - 1)$$

geometry of an ellipse:

- a: semimajor axis, $2a = r_{\min} + r_{\max} = 2p/(1-\epsilon^2)$
- b: semiminor axis, hard to find explicitly so use pythagorean thm
- c: focus to center distance: $2c = r_{\max} - r_{\min} = 2p\varepsilon/(1-\epsilon^2) = 2\varepsilon a$
- $r_{\min} = p/(1+\varepsilon)$
- $r_{\max} = p/(1-\varepsilon)$

1.4.2 The Laplace-Runge-Lenz vector

For a general central force, where $\dot{\mathbf{p}} = f(r)\frac{\mathbf{r}}{r}$, we can show

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p}\times\mathbf{L}) = -mr^2 f(r)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right)$$

in the Kepler problem, where $f(r) = -\frac{k}{r^2}$, this ensures the conservation of the vector

$$\mathbf{A} = \mathbf{p}\times\mathbf{L} - mk\frac{\mathbf{r}}{r}$$

This vector lies in the plane of orbit, and $A^2 = m^2k^2 + 2mE\ell^2$.

1.5 Scattering theory

this is kinda a mess of angle names and things, but the crucial thing to remember is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\pi b\,\mathrm{d}b}{2\pi \sin\theta\,\mathrm{d}\theta} = \frac{b}{\sin\theta}\frac{\mathrm{d}b}{\mathrm{d}\theta}$$

1.6 Rigid Body Motion

three main types of problems:

- torque free motion: can precess, rotation about intermediate axis unstable
- nonzero torque, eg weighted top: precession about vertical, axis of rotation can change (nutation)
- rolling without slipping

components of rotational motion:

- rotation about an axis
- precession of this axis (change in ϕ)
- nutation (change in spherical θ), bobbing as it precesses

basic strategy: decompose motion into CM + movement about the CM

$$\mathbf{F}_{\text{ext}} = M\mathbf{a}_{\text{CM}} \qquad \mathbf{r}_{\text{CM}} \times \mathbf{F}_{\text{ext}} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}_{\text{CM}} = \mathbf{N}_{\text{about CM}}$$
$$\mathbf{L}_{\text{tot}} = \underbrace{\mathbf{L}_{\text{CM}}}_{\mathbf{r}_{\text{CM}} \times M\mathbf{v}_{\text{CM}}} + \mathbf{L}_{\text{about CM}} \qquad T = T_{\text{CM}} + T_{\text{rot}}$$

In general, if a vector $\mathbf{A}$ of fixed length rotates with an angular velocity $\boldsymbol{\omega}$, then

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{A}$$

In particular, when measuring $\mathbf{r}$ from a point $\mathbf{R}$ on a rotating body,

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_f = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_r + \boldsymbol{\omega} \times \mathbf{r}$$

and we typically take p to be the CM. Taking the second derivative,

$$\mathbf{a}_f = \mathbf{a}_r + \underbrace{\ddot{\mathbf{R}}_f}_{\text{translation}} + \underbrace{\dot{\omega} \times \mathbf{r}}_{\text{rotation}} + \underbrace{2\omega \times \mathbf{v}_r}_{\text{coriolis}} + \underbrace{\omega \times (\omega \times \mathbf{r})}_{\text{centrifugal}}$$

Note that if we have angular momentum in a fixed frame, we can look at the torque in the fixed frame by considering $\frac{\mathrm{d}L}{\mathrm{d}t}$ in the body frame; this gives us Euler's equations

1.6.1 Inertia tensor

We define the inertia tensor by

$$I_{jk} = \int \rho(\mathbf{r}) \, (r^2 \delta_{ij} - x_i x_j) \, \mathrm{d}V$$

With $\mathbf{L} = \mathbb{I} \cdot \boldsymbol{\omega}$, kinetic energy is given by

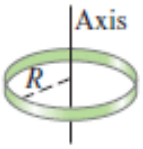
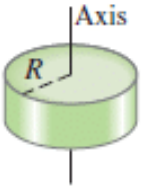
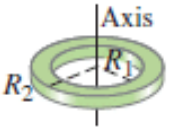
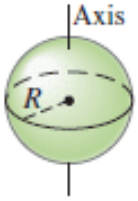
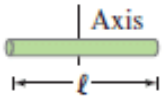
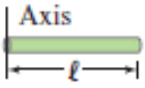
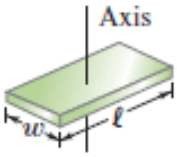
$$T = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbb{I} \cdot \boldsymbol{\omega} = \frac{1}{2} \mathbf{L} \cdot \mathbb{I}^{-1} \cdot \mathbf{L}$$

We can diagonalize $\mathbb{I}$ using principle axes, which in general are time-dependent.

Forget about the parallel axis theorem, but it's good to know the **perpendicular axis theorem:** For a planar object, the sum of the moments of inertia in the plane equals the moment of inertia perpendicular to the plane

$$I_{1,\parallel} + I_{2,\parallel} = I_{\perp}$$

Table of Is for basic shapes (memorize these):

Thin hoop, radius R	Through center		MR^2
Thin hoop, radius R width w	Through central diameter		$\frac{1}{2}MR^2 + \frac{1}{12}Mw^2$
Solid cylinder, radius R	Through center		$\frac{1}{2}MR^2$
Hollow cylinder, inner radius R_1 outer radius R_2	Through center		$\frac{1}{2}M(R_1^2 + R_2^2)$
Uniform sphere, radius R	Through center		$\frac{2}{5}MR^2$
Long uniform rod, length ℓ	Through center		$\frac{1}{12}M\ell^2$
Long uniform rod, length ℓ	Through end		$\frac{1}{3}M\ell^2$
Rectangular thin plate, length ℓ, width w	Through center		$\frac{1}{12}M(\ell^2 + w^2)$

1.6.2 Rotation under torque: the massive top

This is fairly straightforward: construct the lagrangian in terms of three angles: a rotation of the top itself, the precession of the top around in a circle, and a nutation angle that allows the top to bob up and down under the influence of gravity. Introduce the conserved quantities (energy and two conjugate momenta to the rotation and precession angles) and make the substitution $u = -\sin\theta$ for θ the nutation angle, with $\theta = 0$ taken to be horizontal. Then $u = 1$ means the top is upright, $u = 0$ means it's horizontal, and $u = -1$ means it's upside down. We get an equation that looks like $\dot{u}$ = a cubic polynomial, so we have 3 cases for where the roots lie: 3 distinct real roots, a double root and a crossing point, or only one real root.

1.6.3 Rolling without slipping

The rolling without slipping constraint is that *the velocity of the point of contact has to be equal to the velocity of the surface*, where the contact point has motion due to the center of mass and its rotation about the CM, and the surface is usually (but not always!) stationary.

Use Newton's 2nd and the torque equation $r \times F = N = \frac{\mathrm{d}L}{\mathrm{d}t}$. For the torque equation, we can choose torque wrt center of mass, the contact point, or about some arbitrary point (the most useful is usually CM though). You can also solve this with Lagrangians, so long as you can integrate the semiholonomic constraint into one that doesnt depend on the velocities or include it with lagrange multipliers.

Note that the angular momentum can often be weird! For the right hand side, you don't necessarily want to rely on the parallel axis theorem to get I so that you can just use $I\omega$; you should directly find $\mathbf{L} = \mathbf{L}_{\mathrm{CM}} + \mathbf{L}_{\mathrm{rot}}$ and take its derivative. (In particular $\mathbf{L}_{\mathrm{CM}}$ is the weird one; the momentum about the CM is typically fine to think of as $I\omega$)

1.7 Continuous Media

1.7.1 Strings and surfaces

Can solve these using forces: consider a $\mathrm{d}\ell$ or $\mathrm{d}A$ and look at the forces, including tension

Or can solve them variationally, by minimizing the potential energy with a fixed boundary and length and stuff. To include the length constraint, do it with a multiplier. You can often find "energy" conserved by legendre transforming.

It's good to know that waves on strings follow the wave equation with a velocity

$$v = \sqrt{\frac{T\ell}{m}}$$

1.7.2 Fluids

Generally done by looking at forces and energy in a small part of the fluid.

Pressure p is such that $-\nabla p$ acts as a force per unit volume: it's the force on a $\mathrm{d}V$ due to the rest of the fluid. Thus there is a potential energy associated to being surrounded by fluid.

In static fluids, the potential energy is constant, since there is no energy cost to slow mixing. Typically see things like

$$U = p + \psi = \text{const}$$

with ψ the potential energy per unit volume due to external forces (e.g. $\psi = \rho g h$ for gravity, but we can also have centrifugal energy $-\frac{1}{2}\rho\Omega^2 r^2$ or something)

In a flowing fluid with no viscosity, we have a similar energy conservation:

$$p + \psi + \frac{1}{2}\rho u^2 = \text{const} \quad \text{and} \quad \nabla \cdot (\rho \mathbf{u}) = -\frac{\partial \rho}{\partial t}$$

for an incompressible fluid, the density never changes, so we have that the divergence of the velocity field is zero.

If the velocity field is rotationless, we can write it as the gradient of some potential $\mathbf{u} = \nabla\phi$, and incompressibility makes that potential satisfy laplace's equation $\nabla^2\phi = 0$

Viscosity makes it so that layers of fluid will try to pull on the surrounding layers; say we look at $u_x(z)$ for some fluid flowing in the x direction where lower layers move slower than higher layers. We have

$$F_x/A = \eta \frac{\mathrm{d}u_x}{\mathrm{d}z} \quad \text{for some constant } \eta$$

2 E&M

2.1 Maxwells equations and potentials

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \qquad \nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A} \qquad \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \qquad \mathbf{J} = \sigma \mathbf{E}$$

Can get MEs in potential form by subbing in pretty easily.

Gauge-dependent equations In the Lorenz gauge ($\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 V = 0$) and in terms of four-vectors $A^\mu = (V/c,\ \mathbf{A})$ and $J^\mu = (c\rho,\ \mathbf{J})$

$$\left(\epsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} - \nabla^2\right) A^\mu = \mu_0 J^\mu$$

in magnetostatics, we can pick the coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) to get biot savart:

$$\mathbf{A}(x) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{|x - y|} \, \mathrm{d}^3 y$$

Common BCs (from integrating gauss/ampere):

$$\hat{n} \cdot [\mathbf{D}_{\text{out}} - \mathbf{D}_{\text{in}}] = \sigma_f \qquad \hat{n} \times [\mathbf{E}_{\text{out}} - \mathbf{E}_{\text{in}}] = 0$$
$$\hat{n} \cdot [\mathbf{B}_{\text{out}} - \mathbf{B}_{\text{in}}] = 0 \qquad \hat{n} \times [\mathbf{H}_{\text{out}} - \mathbf{H}_{\text{in}}] = \mathbf{K}_f$$

Plane waves in a vacuum

$$\mathbf{E} = \mathbf{E}_0 e^{i(k \cdot r - \omega t)} \qquad \mathbf{B} = \mathbf{B}_0 e^{i(k \cdot r - \omega t)}$$

satisfy maxwell iff

$$\mathbf{E}_0 \perp \mathbf{B}_0 \perp \mathbf{k} \implies \mathbf{B}_0 = \frac{1}{\omega} \mathbf{k} \times \mathbf{E}_0$$

2.2 Energy, momentum, etc

- The energy per unit volume stored in the EM fields is

  $$u = \frac{1}{2}\left(\epsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2\right)$$

  the integral gives the work needed to set up a given charge/current distrib

- The energy per unit time per unit area transported by the fields is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$$

  the surface integral gives the energy flux

- The rate at which work is done on the charges per unit volume is

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \mathbf{E} \cdot \mathbf{J} = -\frac{\mathrm{d}u}{\mathrm{d}t} - \nabla \cdot \mathbf{S}$$

  this gives a continuity equation in the case of no charges/currents

- The force per unit volume done on charges is

$$\mathbf{f} = \frac{\mathrm{d}\mathbf{p}_{\text{mech}}}{\mathrm{d}t} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} = \nabla \cdot \boldsymbol{\sigma} - \epsilon_0\mu_0\frac{\partial \mathbf{S}}{\partial t}$$

  for σ_{ij} the stress tensor, which is the force done in the i direction on a surface oriented in the j direction

$$\sigma_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - u\delta_{ij}$$

  and the last term representing the momentum density stored in the EM fields,

$$\mu_0\epsilon_0\mathbf{S} = \epsilon_0\mathbf{E} \times \mathbf{B}$$

- The angular momentum stored in the fields is the definition you'd expect:

$$\boldsymbol{\ell} = \mathbf{r} \times \mu_0\epsilon_0\mathbf{S} = \epsilon_0\mathbf{r} \times (\mathbf{E} \times \mathbf{B})$$

  Note that static field scan still carry linear/angular momenta!

We combine everybody together into the stress-energy tensor:

$$T^{\mu\nu} = \begin{pmatrix} u & \sqrt{\epsilon_0\mu_0}\mathbf{S} \\ \sqrt{\epsilon_0\mu_0}\mathbf{S} & -\boldsymbol{\sigma} \end{pmatrix}$$

2.3 Dipoles and stuff

$$\mathbf{p} = \int \mathbf{r}\rho(r)\,\mathrm{d}^3\mathbf{r} \qquad U = -\mathbf{p}\cdot\mathbf{E} \qquad \mathbf{N} = \mathbf{p}\times\mathbf{E}$$

$$V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\mathbf{p}}{r^2} \qquad \mathbf{E}_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{r} - \mathbf{p}] = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right]$$

$$\mathbf{Q}_{ij} = \int \rho(\mathbf{r})\Big(3r_i r_j - r^2\delta_{ij}\Big)\,\mathrm{d}^3 r$$

$$\mathbf{m} = \frac{1}{2}\int \mathbf{r}\times\mathbf{J}(r)\,\mathrm{d}^3\mathbf{r} \qquad U = -\mathbf{m}\cdot\mathbf{B} \qquad \mathbf{N} = \mathbf{m}\times\mathbf{B}$$

$$\mathbf{A}_{\text{dip}} = \frac{\mu_0}{4\pi}\frac{1}{r^2}\mathbf{m}\times\hat{\mathbf{r}} \qquad \mathbf{B}_{\text{dip}} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\mathbf{r})\hat{r} - \mathbf{m}] = \frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right]$$

2.4 Fields in matter

$$\nabla\cdot\mathbf{D} = \rho_f \qquad \nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}$$

Constitutive relations relate $\mathbf{D}\leftrightarrow\mathbf{E}$, $\mathbf{H}\leftrightarrow\mathbf{E}$. In linear media,

$$\begin{aligned}\mathbf{D} = \epsilon\mathbf{E} &= \epsilon_0(1+\chi_e)\mathbf{E} &&= \epsilon_0\mathbf{E} + \mathbf{P}\\ \mathbf{B} = \mu\mathbf{H} &= \mu_0(1+\chi_m)\mathbf{H} = \mu_0(\mathbf{H}+\mathbf{M})\end{aligned}$$

We get surface and bound charges and currents due to $\mathbf{P}$ and $\mathbf{M}$:

$$\begin{aligned}\sigma_b &= \mathbf{P}\cdot\hat{\mathbf{n}} & \rho_b &= -\nabla\cdot\mathbf{P}\\ \mathbf{K}_b &= \mathbf{M}\times\hat{\mathbf{n}} & \mathbf{J}_b &= \nabla\times\mathbf{M}\end{aligned}$$

If we have a material with conductivity $\sigma \gg 1$, maxwell + ohm gives

$$\epsilon\mu\partial_t^2\mathbf{E} - \nabla^2\mathbf{E} + \mu\sigma\partial_t\mathbf{E} = 0$$

and assuming a wave ansatz

$$E = \frac{1}{2}\Big(E_0 e^{i(kz-\omega t)} + \text{c.c.}\Big) \implies k^2 = \mu\omega(\epsilon\omega + i\sigma)$$

and we have the dispersion relation for the material

2.4.1 Plasma

The idea here is that instead of using Ohm to rewrite the current, we take $\mathbf{J} = -e\nu\mathbf{v}$ and use newton's second law to say $m_e\partial_t\mathbf{v} = -e\mathbf{E}$. Taking this all together and making our wave ansatz, we get

$$\omega^2 = c^2k^2 + \omega_p^2 \quad \text{for} \quad \omega_p^2 = \frac{e^2\nu}{\epsilon_0 m_e}$$

and we must have a wave with a frequency above the plasma frequency for it to not decay evanescently. We can rewrite the wave equation (after assuming waveform) as

$$-\nabla^2\mathbf{E} - \mu_0 \underbrace{\epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)}_{\epsilon(\omega)}\omega^2\mathbf{E}$$

2.4.2 Waveguides and Cavities

We assume propagation in the z direction along the waveguide C,

$$\mathbf{E}(\mathbf{r}, t) = e^{i(kz-\omega t)}[\mathbf{E}_\perp(\mathbf{r}_\perp) + E_z(\mathbf{r}_\perp)\hat{\mathbf{z}}] \qquad \mathbf{H}(\mathbf{r}, t) = e^{i(kz-\omega t)}[\mathbf{H}_\perp(\mathbf{r}_\perp) + H_z(\mathbf{r}_\perp)\hat{\mathbf{z}}]$$

where the fields are not necessarily transverse! We have nonzero z components of the fields. Plugging into Maxwell and looking only inside the perfect conducting waveguide (where $\rho = \mathbf{J} = 0$) we can assume

$$\nabla \times \mathbf{E} = i\omega\mu\mathbf{H} \quad \text{and} \quad \nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E}$$

and Maxwell tells us (the first is an automatic consequence)

$$\begin{aligned}
\nabla_\perp \cdot \mathbf{E}_\perp &= -ikE_z & \nabla_\perp \cdot \mathbf{H}_\perp &= -ikH_z \\
\nabla_\perp \times \mathbf{E}_\perp &= i\omega\mu H_z\hat{\mathbf{z}} & \nabla_\perp \times \mathbf{H}_\perp &= -i\omega\epsilon E_z\hat{\mathbf{z}} \\
i\omega\mu\mathbf{H}_\perp - ik\hat{\mathbf{z}} \times \mathbf{E}_\perp &= -\hat{\mathbf{z}} \times (\nabla_\perp E_z) & i\omega\epsilon\mathbf{E}_\perp + ik\hat{\mathbf{z}} \times \mathbf{H}_\perp &= \hat{\mathbf{z}} \times (\nabla_\perp H_z)
\end{aligned}$$

taking $\hat{\mathbf{z}}\times$ the bottom equations allows us to determine the transverse $\mathbf{F}_\perp$ from the longitudinal components:

$$\begin{aligned}
\mathbf{H}_\perp &= \frac{ik}{\gamma^2}\nabla_\perp H_z + \frac{i\omega\epsilon}{\gamma^2}\hat{\mathbf{z}} \times \nabla_\perp E_z \\
\mathbf{E}_\perp &= \frac{ik}{\gamma^2}\nabla_\perp E_z - \frac{i\omega\mu}{\gamma^2}\hat{\mathbf{z}} \times \nabla_\perp H_z
\end{aligned}$$

for $\gamma^2 = \mu\epsilon\omega^2 - k^2$. We can show that both transverse fields individually satisfy

$$[\nabla_\perp^2 + \gamma^2]F_z = 0$$

Inside a conducting tube *with simply connected base*, assuming that $E_z = 0$ forces $\mathbf{H}_\perp = 0$ and vice versa. This means that if we assume both are zero, we have no wave propagation. However, other shapes do support these TEM modes.

In general, the strategy is to solve for the F_z and then use this to determine the longitudinal components. Any solution can be written as a linear combination of the three cases.

TEM modes for these waves, $E_z = H_z = 0$. Thus, our $\mathbf{E}_\perp$ and $\mathbf{H}_\perp$ are both harmonic functions of some potential that satisfies Laplace's equation:

$$\nabla_\perp^2 \phi = 0$$

where one of the fields is determined by the other from the last Maxwell equation.

TE modes in this case, $E_z = 0$. Thus,

$$[\nabla_\perp^2 + \gamma^2]H_z = 0 \qquad \mathbf{H}_\perp = \frac{ik}{\gamma^2}\nabla_\perp H_z \qquad \mathbf{E}_\perp = -\frac{i\omega\mu}{\gamma^2}\hat{\mathbf{z}} \times \nabla_\perp H_z$$

Where our boundary condition is that $\partial_{\mathbf{n}}\mathbf{H}_z$ is zero on the boundary.

TM modes in this case, $H_z = 0$. Thus,

$$[\nabla_\perp^2 + \gamma^2]E_z = 0 \qquad \mathbf{H}_\perp = \frac{i\omega\epsilon}{\gamma^2}\hat{\mathbf{z}} \times \nabla_\perp E_z \qquad \mathbf{E}_\perp = \frac{ik}{\gamma^2}\nabla_\perp E_z$$

Where our boundary condition is that $\mathbf{E}_z$ is zero on the boundary.

Cavities this is a waveguide that is truncated to be of finite length, so now we need to consider the BCs on the end plates, which read

$$\mathbf{H}_z = 0 \qquad \mathbf{E}_\perp = 0$$

to solve this, treat it as an infinite waveguide and take linear combinations to get standing-wave type solutions that are zero on the plates.

2.5 Laplace's equation and Multipole expansion

2.5.1 2D Laplace

In two dimensions, after separation of variables, we find that both X and Y satisfy harmonic oscillator equations with the property that $k_x^2 + k_y^2 = 0$:

$$X''(x) = k^2 X \qquad Y''(y) = -k^2 Y$$

this means in one dimension we'll have trig solutions and in the other we'll have exponentials. Pick whatever fits your BCs. You may need to add together solutions for a number of ks; often this comes from using a fourier series to match the trig dimension to some boundary potential – to find coefficients here, use the orthogonality of the trig functions:

$$\int_0^a \sin(n\pi y/a)\sin(m\pi y/a)\,\mathrm{d}y = \frac{a}{2}\delta_{mn}$$

2.5.2 3D Laplace

Cartesian coords Separation of variables gives us

$$X'' = (k^2 + \ell^2)X \qquad Y'' = -k^2 Y \qquad Z'' = -\ell^2 Z$$

so we again get some mix of exponentials and trigs depending on what makes the most physical sense

Spherical coords

$$\nabla^2 V = 0 \implies V(\mathbf{r}) = Y_{\ell m}(\theta,\,\phi)\left[A_{\ell m} r^{\ell} + B_{\ell m} r^{-(\ell+1)}\right]$$

in the case of cylindrical symmetry, $m = 0$ and the equation simplifies to

$$V = P_\ell(\cos\theta)\left[A_\ell r^{\ell} + B_\ell r^{-(\ell+1)}\right]$$

2.6 Method of images

Idea: replace a conductor by point charges that we know how to deal with; we go from a boundary value problem to an equivalent setup that matches the BCs (keep in mind that it is just that, though – a boundary value problem). You can do this with charges or with magnetic dipoles depending on what you need.

2.7 Capacitance and inductance

$$C = \frac{Q}{\Delta V} = -\epsilon_0 \int_{\partial\Omega} \mathrm{d}S \cdot \nabla\phi \qquad \text{for } \phi = 1 \text{ in } \Omega,\ 0 \text{ at } \infty$$

$$U_{\text{cap}} = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$$

Strategy: assume a charge $\pm Q$ on the parts, calculate the potential difference or energy content, use this to get the capacitance.

A loop of current will generate a magnetic field. The flux of that field through a different loop is

$$\Phi_2 = \int_2 \mathbf{B}_1 \cdot \mathbf{S}_2 = M_{21} I_1$$

where M_{21} is the mutual inductance and is symmetric. You can also calculate self inductance, relating the flux through a loop to the current in the loop producing it.

$$\Phi = LI \qquad U = \frac{1}{2}LI^2$$

also dont forget maxwell tells us induced EMF:

$$\mathcal{E} = -L\frac{\mathrm{d}I}{\mathrm{d}t}$$

2.8 Radiation

We solve ME in Lorenz gauge using the retarded greens function

$$\left(\epsilon_0\mu_0\frac{\partial^2}{\partial t^2} - \nabla^2\right)A^\mu = \mu_0 J^\mu = \mu_0\, q\, u^\mu(t)\delta(\mathbf{x} - \mathbf{x}_0(t))$$

where $u^\mu = (c,\ \mathbf{v})$ is the four velocity of the particle. Then

$$A^\mu(\mathbf{x},\ t) = \frac{\mu_0 q}{4\pi}\frac{u^\mu(\tau)}{|\mathbf{x} - \mathbf{x}_0(\tau)|}\frac{1}{1 - \boldsymbol{\beta}(\tau)\cdot\mathbf{n}(\tau)} \qquad \tau = t - \frac{1}{c}|\mathbf{x} - \mathbf{x}_0(\tau)|$$

where $\boldsymbol{\beta} = \mathbf{v}/c$ and $\mathbf{n} = \frac{\mathbf{x}-\mathbf{x}_0(\tau)}{|\mathbf{x}-\mathbf{x}_0(\tau)|}$ is the unit vector pointing from the source to the point of measurement. This is the Lienard-Wiechert potential. The exact expressions of the fields can be derived from this. We approximate:

1. $|\mathbf{x}| \gg 1$, meaning we can drop $1/|x - x_0|^2$s

2. $\beta \ll 1$, meaning we can drop $(n \cdot \beta)$

which gives fields

$$\mathbf{E}(\mathbf{x},\, t) = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}}(\tau))}{c|\mathbf{x}|} \quad \text{and} \quad \mathbf{B}(\mathbf{x},\, t) = \frac{1}{c}\mathbf{n}\times\mathbf{E}(\mathbf{x},\, t)$$

The Poynting vector is

$$\mathbf{S}(\mathbf{r}) = \frac{E^2}{\mu_0 c}\mathbf{n} = \frac{\mu_0 q^2}{16\pi^2 c}\frac{a^2(\tau)\sin^2\theta}{|\mathbf{r}|^2}\hat{r}$$

with the θ measured from the direction of acceleration. Integrating gives Larmor,

$$P = \frac{\mu_0 q^2 a^2(\tau)}{6\pi c}$$

2.8.1 Multipole expansion

Define the radiation vector $\boldsymbol{\alpha}$ by

$$\frac{\partial \mathbf{A}_{\text{rad}}(\mathbf{r},\, t)}{\partial t} = \frac{\mu_0}{4\pi r}\boldsymbol{\alpha}(\mathbf{r},\, t) \implies \boldsymbol{\alpha}(\mathbf{r},\, t) = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\mathbf{r}'\,\mathbf{J}\left(\mathbf{r},\, t - \frac{r + \hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)$$

then the fields and power are given by

$$c\mathbf{B}_{\text{rad}}(\mathbf{r},\, t) = -\frac{\mu_0}{4\pi r}\hat{\mathbf{r}}\times\boldsymbol{\alpha} \quad \text{and} \quad \mathbf{E}_{\text{rad}}(\mathbf{r},\, t) = -\hat{\mathbf{r}}\times c\mathbf{B}_{\text{rad}}$$
$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{\mu_0}{16\pi^2 c}|\hat{\mathbf{r}}\times\boldsymbol{\alpha}(\mathbf{r},\, t)|^2$$

The idea here is to expand $\mathbf{J}$:

$$\mathbf{J}\left(\mathbf{r},\, t - \frac{r + \hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right) \approx \sum_{n=0}\frac{1}{n!}\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\frac{\partial}{\partial t}\right)^n \mathbf{J}(\mathbf{r},\, t - r/c)$$

so that

$$\boldsymbol{\alpha}(\mathbf{r},\, t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{p}(t - r/c) + \frac{1}{c}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{m}(t - r/c) + \frac{1}{c}\frac{\mathrm{d}^3}{\mathrm{d}t^3}\mathbf{Q}(t - r/c)\cdot\hat{\mathbf{r}} + \ldots$$

This expansion gives us the usual fields for electric dipole, magnetic dipole, etc.

3 Quantum Mechanics

3.1 Postulates and Principles of QM

Really basic stuff:

1. Physically realizable states live in projective hilbert space
2. Observables $\iff$ hermitian operators on this HS
3. Possible observations given by eigvals of these ops
4. $P(a_n) = |\langle a_n|\psi\rangle|^2$, after measurement $|\psi\rangle \to \hat{P}_{a_n}|\psi\rangle = |a_n\rangle$ where the last equality holds if the eigval is nondegenerate. $\langle f(A)\rangle = |\langle a_i|\psi\rangle|^2 f(a_i)$
5. $i\hbar\partial_t|\psi\rangle = \hat{H}|\psi\rangle$, where eigenfunctions $|\psi_n\rangle$ satisfy $\psi_n(x,\,t) = \psi_n(x)\exp(-iE_nt/\hbar)$ and an arbitrary vector can be written as a linear combination of these steady states.
6. n identical particles are always in states which are eigenfunctions of all exchange operators – eigval of ± 1 depending on statistics.

kernel of fourier transform used to move between x and p representations:

$$\langle x|p\rangle = (2\pi\hbar)^{-1/2}\exp(-ixp/\hbar)$$
$$\psi(p) = \langle p|\psi\rangle = \int \mathrm{d}x\,\langle p|x\rangle\,\langle x|\psi\rangle = \int \frac{\mathrm{d}x}{\sqrt{2\pi\hbar}} e^{+ipx/\hbar}\psi(x)$$

important distinction:

- expected location: $\langle\hat{r}\rangle$
- most probable location: maximum of $|\psi|^2$
- most probable radius: r that maximizes $\int \mathrm{d}\Omega\, r^2|\psi|^2$

probability current:

$$j = \frac{\hbar}{2mi}[\psi^*(\partial_x\psi) - (\partial_x\psi)^*\psi] = \frac{\hbar}{m}\,\mathrm{Im}(\psi^*\partial_x\psi)$$

3.2 Operators and Pictures

$$\left\langle \hat{A} \right\rangle = \langle \psi | \hat{A} | \psi \rangle \qquad \frac{\mathrm{d}}{\mathrm{d}t} \left\langle \hat{A} \right\rangle = \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle + \frac{i}{\hbar} \left\langle [\hat{H},\, \hat{A}] \right\rangle$$

$$\Delta A = \sqrt{\left\langle (\hat{A} - \left\langle \hat{A} \right\rangle)^2 \right\rangle} = \sqrt{\left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2} \qquad \Delta A \Delta B \geq \frac{1}{2} | \left\langle [A,\, B] \right\rangle |$$

Eherenfest derivable from just performing the derivative (being sure to hit the $|\psi\rangle$s) and the uncertainty relation derivable from the schwartz inequality on $\hat{O} - \left\langle \hat{O} \right\rangle$

	Heisenberg	Dirac	Schrodinger
States	constant	evolve with V	evolve with H
Operators	evolve with H	evolve with H_0	constant

where operators evolve as $-i\hbar\partial_t \hat{A} = [\bullet,\, \hat{A}]$
and states evolve as $i\hbar\partial_t |\psi\rangle = \bullet\, |\psi\rangle$

3.3 Simple Harmonic Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 = \hbar\omega \left(\frac{\hat{P}^2 + \hat{X}^2}{2} \right)$$

after adimensionalizing the operators:

$$\hat{P} = \frac{\hat{p}}{\sqrt{m\hbar\omega}} \quad \text{and} \quad \hat{X} = \hat{x} \sqrt{\frac{m\omega}{\hbar}} \quad \text{for which} \quad [\hat{X},\, \hat{P}] = i$$

check: $\hbar\omega$ is an energy, $[mE] = [m]^2[c]^2 = [p]^2$, similarly for the $\hat{x}$ prefactor

3.3.1 Ladder operators

Nice in terms of adim ops:

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{X} + i\hat{P}) \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{X} - i\hat{P})$$

$$[\hat{H},\, \hat{a}] = -\hbar\omega\hat{a} \qquad [\hat{H},\, \hat{a}^\dagger] = +\hbar\omega\hat{a}^\dagger \qquad [\hat{a},\, \hat{a}^\dagger] = 1$$

$$H = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \quad \text{and} \quad \hat{a}^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle\,,\ \hat{a}\, |n\rangle = \sqrt{n}\, |n-1\rangle$$

working in the dimensionless spatial coordinates X and $-i\partial_X$, it's not too bad to work out wavefunctions starting from

$$\psi_0(X) = \langle X|0\rangle = N_0 e^{-X^2/2} \qquad \psi_n(X) = H_n(X)\psi_0(X)$$

3.3.2 Coherent states

indexed by $\alpha \in \mathbb{C}$ – this is their eigenvalue under $\hat{a}$; set initial position and velocity with the two dof:

$$\left\langle \hat{X} \right\rangle = \sqrt{2}\,\mathrm{Re}[\alpha] \qquad \left\langle \hat{P} \right\rangle = \sqrt{2}\,\mathrm{Im}[\alpha]$$

as time goes on, they stay coherent but α evolves (see eherenfest)

$$\begin{aligned} |\alpha\rangle &= e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} |0\rangle \\ &= \exp\left(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right) |0\rangle \\ &= \exp\left(i\sqrt{2}(\hat{X}\,\mathrm{Im}[\alpha] - \hat{P}\,\mathrm{Re}[\alpha])\right) |0\rangle \end{aligned}$$

can use BCH to split the exp into two; recall

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]+\ldots}$$

3.4 Angular momentum and spin

Take some operators $\hat{J}_i$ that satisfy $SU(2)$ commutation $[\hat{J}_i, \hat{J}_k] = i\hbar\epsilon_{ijk}\hat{J}_k$

we pick a basis $|jm\rangle$ that simultaneously diagonalizes $\hat{J}^2$ and $\hat{J}_z$:

$$\hat{J}^2 |jm\rangle = \hbar^2 j(j+1) |jm\rangle \qquad \hat{J}_z |jm\rangle = \hbar m |jm\rangle$$

these imply j is (nonnegative) integer or half-integer and $m \in [-j, j]$

ladder between the ms with

$$\hat{J}_\pm = \hat{J}_x \pm \hat{J}_y \qquad [\hat{J}_z, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm \qquad [\hat{J}^2, \hat{J}_\pm] = 0$$

for normalized $|jm\rangle$ states,

$$\hat{J}_\pm |jm\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j,\, m\pm 1\rangle$$

remember signs: we kill the extreme m states

spherical harmonics, eigs of the orbital angmom $\hat{L}^2$, only get integer js (ℓs)

for spin (internal dof) half integer j is okay.

for a fixed j the ms give us $2j+1$ states to work with

3.4.1 Addition: Clebsh-Gordon coefficients

say we have two particles with $|j_1 m_1,\ j_2 m_2\rangle$ in which J_1^2, J_{1z}, J_2^2, J_{2z} are diagonalized. we can also consider $|jmj_1j_2\rangle$ in which J^2, J_z, J_1^2, J_2^2 diagonalized. the obstruction comes about because J^2 and J_{iz} arent simultaneously diagonalizable.

usually we fix j_1 and j_2 (e.g. fixing total spins of particles which are determined by species) and then try to find $|jm\rangle$ in terms of $|m_1 m_2\rangle$. in this case, j can range from $|j_1 - j_2|$ to $j_1 + j_2$. We start with max j and max m and then ladder down, since the ladders for the total is the sum of the individual ladders. we now know $|j,\ m = j - 1\rangle$, so we find a state orthogonal to it to use as $|j - 1,\ m = j - 1\rangle$

Example: $j_1 = \ell = 1$ and $j_2 = s = 1/2$.

$$\left|j\tfrac{3}{2},\ m\tfrac{3}{2}\right\rangle = \left|m_\ell 1,\ m_s\tfrac{1}{2}\right\rangle$$

$$\left|j\tfrac{3}{2},\ m\tfrac{1}{2}\right\rangle = N(L_- + S_-)\left|m_\ell 1,\ m_s\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|m_\ell 0,\ m_s\tfrac{1}{2}\right\rangle + \tfrac{1}{\sqrt{3}}\left|m_\ell 1,\ m_s - \tfrac{1}{2}\right\rangle$$

$$\left|j\tfrac{1}{2},\ m\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}\left|m_\ell 0,\ m_s\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|m_\ell 1,\ m_s - \tfrac{1}{2}\right\rangle$$

these are written in CG tables, which give j, m on each column and m_1, m_2 on each row. the coefficients are given squared, so you need to take the square root before using them.

3.5 Identical particles

this is pretty straightforward: if the particles are indistinguishable, we have statistics effects (fermion or boson). keep in mind that if the particles have a spin and a spatial wavefunction then one part of it might already take care of the (anti)symmetry that you need!

3.6 3D quantum mechanics

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\hat{L}^2}{2mr^2} + V$$

$$\hat{L} = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

gives us solutions of the form

$$\psi_{n\ell m} = R_{n\ell}(r)Y_{\ell m}(\theta,\ \phi)$$

Angular part: the spherical harmonics are eigens of both $\hat{L}^2$ and $\hat{L}_z$ with

$$\hat{L}^2 Y_{\ell m} = \hbar^2 \ell(\ell+1) Y_{\ell m} \quad \text{and} \quad \hat{L}_z Y_{\ell m} = \hbar m Y_{\ell m}$$

up to normalization, they look like

$$Y_{\ell m} = e^{im\phi} P_\ell^m(\cos\theta)$$

with the $\ell = 0$ and $\ell = 1$ ones given explicitly as:

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \qquad Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \qquad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_{1-1} = \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}$$

it might be handy to know

$$\sin\theta e^{i\phi} = \frac{x+iy}{r} \qquad \cos\theta = \frac{z}{r} \qquad \sin\theta e^{-i\phi} = \frac{x-iy}{r}$$

which helps explain the extra $\sqrt{2}$ on the 1 ± 1s.

Radial part: with the ℓ fixed on the sphharm side, our equation reads

$$-\frac{\hbar^2}{2m}\frac{1}{r}\partial_r(r^2 \partial_r R_{n\ell}) + \left[V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}\right] R_{n\ell} = E_{n\ell} R_{n\ell}$$

where the trick is to swap to $u = rR$ where the equation reads

$$\left[-\frac{\hbar^2}{2m}\partial_r^2 + V + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}\right] u_{n\ell} = E_{n\ell} u_{n\ell}$$

which is like TISE with hard well at $x = 0$ (u must be zero when $r = 0$)

3.6.1 Hydrogen

Take V to be Coulomb:

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}$$

then the bound states are ordered by an $n = 1, 2, \ldots$ and $\ell = 0, \ldots, n-1$.

Energy independent of ℓ (and goes like α^2):

$$E_{n\ell} = E_n = \frac{-13.6\text{eV}}{n^2} = -\frac{1}{n^2}\frac{\mu Z^2 e^4}{2(4\pi\epsilon_0)^2\hbar^2}$$

A few wavefuntions:

$$\psi_{100} = \frac{e^{-r/a_0}}{\sqrt{\pi a_0^3}} \qquad \psi_{200} = \frac{e^{-r/2a_0}}{4\sqrt{2\pi a_0^3}}\left[2 - \frac{r}{a_0}\right]$$
$$\psi_{210} = \frac{e^{-r/2a_0}}{4\sqrt{2\pi a_0^3}}\frac{r}{a_0}\cos\theta \qquad \psi_{21\pm1} = \frac{e^{-r/2a_0}}{8\sqrt{\pi a_0^3}}\frac{r}{a_0}\sin\theta e^{\pm i\phi}$$

Relativistic corrections a fine structure correction (energy goes like α^4): take next term in relativistic kinetic energy expansion and perturb hydrogen's degenerate states (L^2, L_z commute with p^2 and thus with the p^4 terms we get), we have a nice basis to perturb with. end up needing some $\langle r^{-n}\rangle$s; the $n = 1$ you can get with virial theorem, and the $n = 2$ with feynman-hellmann theorem. end up getting a perturbation that depends on n and ℓ.

Sporbit coupling fine structure: if $\ell \neq 0$ then the orbital angular momentum couples to the spin of the electron and affects the energy depending on aligned/antialigned. kramer's relation gives further ns from above.

Lamb shift (energy goes like α^5) QED vacuum perturbs the atom

Spin-spin coupling hyperfine structure (energy goes like $\alpha^8 m_e/m_p$: coupling between the spin of the electron and the proton

3.7 Approximation methods

3.7.1 TIPT

Use: we have a hammy that looks like $H = H_0 + \lambda V$ for some small parameter λ where we know how to solve H_0

No degeneracy:

$$E_n^{(1)} = \langle \psi_n^{(0)} | V | \psi_n^{(0)} \rangle \equiv V_{nn}$$
$$\psi_n^{(1)} = -\sum_{m \neq n} \frac{\psi_m^{(0)} V_{mn}}{E_m^{(0)} - E_n^{(0)}}$$
$$E_n^{(2)} = -\sum_{m \neq n} \frac{|V_{mn}|^2}{E_m^{(0)} - E_n^{(0)}}$$

Degeneracy: pick a basis for the degenerate subspace of H_0 such that V is diagonalized (e.g. if we know eigenstates of V already that H_0 doesnt care about); these diagonal elements are the first-order corrections to the energy. in this basis, the second-order corrections are given by the same sum, just with a restriction on the sum so that it doesnt blow up.

3.7.2 TDPT

Idea: invert the schrodinger equation and write

$$\begin{aligned}\psi(t) &= \mathcal{T} \exp\left(-\frac{i}{\hbar} \int_0^t H(t') \, \mathrm{d}t'\right) \psi(0) \\ &= \psi(0) - \frac{i}{\hbar} \int_0^t H(t_1) \, \mathrm{d}t_1 \, \psi(0) - \frac{1}{\hbar^2} \int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 \, H(t_2) H(t_1) \psi(0) + \dots\end{aligned}$$

(note: the bounds on the inner integrals eat the $1/n!$ from the expansion and preserve the time-ordering)

we typically do this in the interaction picture, so ψ evolves with just the perturbation V, and we're also typically interested in transitions between eigenstates of H_0. these assumptions give us the matrix element

$$\mathcal{M}_{i \to f} = -\frac{i}{\hbar} \int_0^t \mathrm{d}t \exp\big(i(E_f - E_i)t/\hbar\big) \, \langle f | V(t) | i \rangle$$

and probability of transition $|\mathcal{M}|^2$

3.7.3 Fermi's golden rule

$$P_{i\to f} \approx \frac{2\pi t}{\hbar}|\langle f|F|i\rangle|^2\delta(E_f - E_i - \hbar\omega) + \frac{2\pi t}{\hbar}|\langle f|F^\dagger|i\rangle|^2\delta(E_f - E_i + \hbar\omega)$$

$$\Gamma = \frac{\mathrm{d}P}{\mathrm{d}t} = \frac{2\pi}{\hbar}|\langle F\rangle|^2\delta(\Delta E - \hbar\omega) + \frac{2\pi}{\hbar}|\left\langle F^\dagger\right\rangle|^2\delta(\Delta E + \hbar\omega)$$

comments:

- often take a $\langle f|$ continuum with density of states $\rho(E_f)$, i.e. $\rho(E_f)\,\mathrm{d}E$ is the number of accessible states with energy $\in [E_f,\ E_f + \mathrm{d}E_f]$. We basically then just swap delta for this rho:

$$\Gamma_{\text{tot}} = \frac{2\pi}{\hbar}\rho(E_i + \hbar\omega)\left|\left\langle f_{(E_i+\hbar\omega)}\middle|F\middle|i\right\rangle\right|^2 + \frac{2\pi}{\hbar}\rho(E_i - \hbar\omega)\left|\left\langle f_{(E_i-\hbar\omega)}\middle|F^\dagger\middle|i\right\rangle\right|^2$$

- if $\omega = 0$, we must take

$$\Gamma = \frac{2\pi}{\hbar}|\langle f|V|i\rangle|^2\delta(E_i - E_f)$$

assumptions:

- $V(t) = Fe^{-i\omega t} + F^\dagger e^{i\omega t}$
- "t large" in that $\rho(E_f)|\langle F\rangle|^2$ shouldnt change very much for the energy range $\hbar/t$; we need the time to be large relative to [$\hbar$ over the characteristic energy scale for $\rho(E_f)$ to vary]
- "t not too large" in that there must be many states available within the energy scale; we need the time to be small relative to [$\hbar$ over the characteristic energy spacing] for the delta function to make sense

$$\frac{\hbar}{E_{\text{typical}}} \ll t \ll \frac{\hbar}{\varepsilon_{\text{spacing}}}$$

3.7.4 Variational principle

Use: finding ground state energy

It is a fact that

$$E_0 \le \frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle}$$

for any wavefunction ψ. thus, we create a family of $\psi_\sigma(x)$ parameterized by some σ (e.g. width of a gaussian or something) and minimize over σ to get a best estimate for E_0

3.7.5 WKB approximation

Use: this is a semiclassical $\hbar\left|\frac{\mathrm{d}p}{\mathrm{d}x}\right|^2 \ll |p(x)|^2$ approximation.

for a slowly-varying $V(x)$, we extract $p(x)$ as if it were classical:

$$p(x) = \sqrt{2m(E - V(x))} \implies \psi(x) \approx \frac{A}{\sqrt{p(x)}} \exp\left(\pm\frac{i}{\hbar}\int_0^x \mathrm{d}y\, p(y)\right)$$

this works best in the cases where:

- $p(x)$ real $\iff E > V$ — "classical region" of oscillatory ϕ
- $p(x)$ imag $\iff E < V$ — "forbidden region" exp growth/decay

to patch together in the intermediate regions, try to match typical boundary conditions and the continuity of the phase.

3.8 Scattering

Note: we also use TDPT in 3D scattering but i put it in the approximation methods section. FGR holds for weak scattering potentials

3.8.1 1D Scattering

Let $V_I = \min[V(-\infty), V(+\infty)]$ and V_{II} be the max. Bound states are discrete and require $\inf V(x) < E < V_I$. We have one scattering state for each $E \in [V_I, V_{II}]$ and two scattering states for each $E > V_{II}$

Take ψ as wave from $-\infty$ being reflected and transmitted:

$$\begin{aligned}\psi(x \to -\infty) &= \underbrace{Ae^{ik_L x}}_{\text{incoming}} + \underbrace{Be^{-ik_L x}}_{\text{outgoing}} \\ \psi(x \to +\infty) &= Ce^{ik_R x}\end{aligned}$$

we have coefficients

$$R = \frac{|B|^2}{|A|^2} \quad \text{and} \quad T = 1 - R = \frac{k_R|C|^2}{k_L|A|^2}$$

we typically use boundary conditions

- continuity of ψ
- continuity of ψ'

however, if we have an infinite potential, e.g. $V = \lambda\delta(x)$, we dont have the latter (which is the condition for differentiability). Instead, we integrate both sides of the SE to get

$$-\frac{\hbar^2}{2m}[\psi'_R - \psi'_L] + \lambda\psi(0) = 0$$

3.8.2 Born Approximation

Easiest derivation: start with FGR, scattering in a periodic box

$$\psi_i(\mathbf{r}) = L^{-3/2}e^{i\mathbf{k}_i\cdot\mathbf{r}} \quad \text{to} \quad \psi_i(\mathbf{r}) = L^{-3/2}e^{i\mathbf{k}_f\cdot\mathbf{r}}$$

where $\mathbf{k}$ is quantized as $2\pi\mathbf{n}/\hbar$. Replacing the delta in FGR with the dos

$$\rho(E)\,\mathrm{d}E = n^2\,\mathrm{d}n\,\mathrm{d}\Omega \implies \rho(E) = \left(\frac{L}{2\pi\hbar}\right)^3 m\sqrt{2mE}\,\mathrm{d}\Omega$$

allows us to calculate

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\omega} = \frac{\mathrm{d}P}{\mathrm{d}t\cdot A}\frac{A}{\mathrm{d}\Omega} = j_{\text{inc}}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$$

and thus

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left|-\frac{m}{2\pi\hbar^2}\int e^{i(\mathbf{k}_i-\mathbf{k}_f)\cdot\mathbf{r}}V(r)\,\mathrm{d}^3\mathbf{r}\right|^2}$$

which is formula obtained via the born approximation.

the actual approximation itself is a bit more complex. we turn the TISE into the helmholtz equation and find a greens function solution that matches our BCs of outgoing waves. we get that the final wavefunction depends on an integral over the final wavefunction, so we plug back in to itself and get an interated integral that we truncate to first order in V and then compare to the asymptotic free-plus-scattered expected form

$$\psi = \psi_{\text{in}} + \psi_{\text{scattered}} \approx e^{i\mathbf{k}_i\mathbf{r}} + \frac{e^{i\mathbf{k}_f\,\mathbf{r}}}{r}f(\theta,\,\phi) \quad \text{where} \quad \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta,\,\phi)|^2$$

if V is spherically symmetric, we can go ahead and do these integrals:
let $\boldsymbol{\kappa} = \mathbf{k}_i - \mathbf{k}_f$ so that $\kappa = 2k\sin\theta/2$ and

$$f(\theta,\,\phi) = -\frac{2m}{\hbar^2\kappa}\int_0^\infty \mathrm{d}\bar{r}\,\bar{r}V(\bar{r})\sin(\kappa\bar{r})$$

3.8.3 Partial wave expansion

recall the 3D spherical SE

$$\left[-\frac{\hbar^2}{2m}\partial_r^2 + V + \frac{\hbar^2\ell(\ell+1)}{2mr^2}\right]u_{n\ell} = E_{n\ell}u_{n\ell}$$

In the far field, we've been ignoring angular momentum since it perturbs the effective potential as r^{-2}. What if we stop ignoring it? then $\psi \sim j_\ell(kr)Y_{\ell m}$ or $\sim n_\ell(kr)Y_{\ell m}$ (bessels and von neumans), which go like sincs and cosincs respectively; nice for bound states but not for scatterings. We instead look at hankels $h_\ell^{(1)} = j_\ell + in_\ell$ which asymptotically look like out- and in-going spherical waves

$$h_\ell^{(1)} \sim \frac{(-i)^{\ell+1}}{x}e^{ix} \qquad h_\ell^{(2)} \sim \frac{(i)^{\ell+1}}{x}e^{-ix}$$

the strategy is to write the incoming plane wave in terms of hankels, which corresponds to different angular momenta. we match these to the large-r boundary conditions to solve for u_ℓ. Taking r to infinity then puts it back into a form where we can recognize $f(\theta)$

this is nice because in the low-energy ($kr_* \ll 1$) limit, we can just take the first ℓ, called S-wave scattering:

1. for $r > r_*$, we take the far field and write $\psi = Ae^{ikz} + B\frac{e^{ikr}}{r}$
2. for $r \gtrsim r_*$, we use the approximation: $\psi \sim A + \frac{B}{r}$
3. for $r < r_*$, we can no longer ignore V: solve $(-\frac{\hbar^2}{2m}\nabla^2 + V)\psi \approx 0$ with the boundary condition at r_*

3.9 Particles in EM fields

$$H = \frac{(p - qA)^2}{2m} + q\phi$$

common situations:

- uniform, time-independent B: usually want to choose a rotationally symmetric $A = \frac{1}{2}B \times r$, eg $A = B/2(-y,\, x,\, 0)$ or $A = B(0,\, x,\, 0)$
- time independent E: just say $E = -\nabla\phi$ as usual and work with the hammy

keep in mind ϕ is not gauge invariant:

$$A \to A + \nabla\chi,\ \phi \to \phi - \partial_t \chi \implies \phi \to e^{iq\chi/\hbar}\phi$$

the current density also changes:

$$j = \frac{1}{m}\operatorname{Re}[\psi^*(p - qA)\psi] = \frac{\hbar}{m}\operatorname{Im}[\psi^*(\nabla - \frac{iq}{\hbar}A)\psi]$$

3.9.1 Zeeman effect

put the Hydrogen atom in a B field; take $A = \frac{1}{2}B \times r$ and the hammy becomes

$$H = \frac{p^2}{2m} - \frac{e^2}{4\pi\epsilon_0} + \underbrace{\frac{e}{2m}B \cdot L}_{\text{cross term}} + \underbrace{\frac{e^2}{8m}|B \times r|^2}_{A^2 \text{ term}}$$

we usually take the B to be small so we can ignore A^2 and just perturb the angular momentum coupling

3.9.2 Landau levels

take a B field in the z direction and let $\mathbf{A} = Bx\hat{\mathbf{y}}$. we can coax this into having plane waves in y and harmonic oscillator in x

3.9.3 Other situations

Aharanov Bohm effect: phase shift due to A
particle on ring with B field through it
spin in B field; $H \sim B \cdot m$
stark effect, which is like zeeman with E and antisemitism

4 Statistical Mechanics

4.1 classical thermo

laws:

0. equilibrium is transitive (gives us idea of temperature)

1. amount of work required to change isolated system's state is independent of how work is performed. for nonisolated systems, change of energy includes a heat term $\Delta E = Q + W$

2. entropy increases

Kelvin: no process exists whose sole effect is to extract heat from a resevoir and turn it into work

Clausius: no process exists whose sole effect is to transfer heat from cold to hot

3. not really as important as the others, but it's

$$\lim_{T \to 0} S(T) = 0$$

- the ground state entropy shouldnt grow extensively
- heat capacities tend to zero

4.1.1 Carnot

isothermal expansion at hot temperature → adiabatic expansion (cools) → isothermal contraction at cold temperature → adiabatic contraction (heats up)

$$\eta = \frac{W}{Q_H} = \frac{Q_H - Q_C}{Q_H} = 1 - \frac{Q_C}{Q_H} \overset{\text{Carnot}}{=} 1 - \frac{T_C}{T_H}$$

4.1.2 entropy

$$S = \int \frac{\delta Q}{T}$$

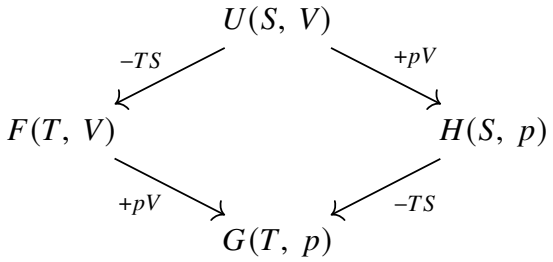
is a function of state – doing the integral in a (REVERSIBLE) loop gives you zero, otherwise we have clausius inequality

$$\oint \frac{\delta Q}{T} \leq 0$$

(tells us that entropy of an isolated system never decreases)

4.1.3 Thermodynamic potentials

at fixed energy, the entropy doesnt decrease. other extremization principles follow:

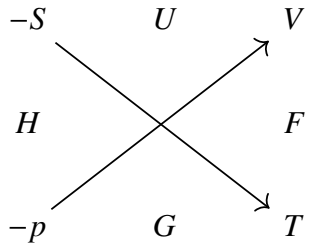


free energy is a measure of the amount of energy free to do work at a finite temperature – at constant temp and volume, free energy can never increase

at fixed temperature and pressure, minimize G.

Note: by extensivity, $G(p,\ T,\ N) = \mu(p,\ T)N$; cf. $\Phi = -p(T,\ \mu)V$

at fixed energy and pressure, consider enthalpy



4.1.4 Maxwell relations

rewrite derivatives that you dont know in terms of things you do!

when looking for something of the form

$$\left.\frac{\partial A}{\partial B}\right|_C$$

the idea is to find A as a first derivative of some function of state that has $\mathrm{d}B$ and $\mathrm{d}C$ as differentials; this lets us swap A for the B derivative. more explicitly,

find a thermodynamic potential of the form $\mathrm{d}X = A\,\mathrm{d}\alpha + \beta\,\mathrm{d}B + \gamma\,\mathrm{d}C$. Then

$$\frac{\partial^2 X}{\partial B \partial \alpha} = \left.\frac{\partial A}{\partial B}\right|_{C,\alpha} = \left.\frac{\partial \beta}{\partial \alpha}\right|_{C,B}$$

as an example, consider

$$\left.\frac{\partial \mu}{\partial p}\right|_T$$

our function of state is

$$\mathrm{d}G = -S\,\mathrm{d}T + V\,\mathrm{d}p + \mu\,\mathrm{d}N \implies \frac{\partial^2 G}{\partial p \partial N} = \left.\frac{\partial \mu}{\partial p}\right|_{N,T} = \left.\frac{\partial V}{\partial N}\right|_{p,T}$$

Heat capacities this does nice things for them; recalling

$$C_\bullet = T\left.\frac{\partial S}{\partial T}\right|_\bullet$$

we find

$$\left.\frac{\partial C_V}{\partial V}\right|_T = T\left.\frac{\partial^2 p}{\partial T^2}\right|_V \quad \text{and} \quad \left.\frac{\partial C_p}{\partial p}\right|_T = -T\left.\frac{\partial^2 V}{\partial T^2}\right|_p \implies C_p - C_V = T\left.\frac{\partial V}{\partial T}\right|_p \left.\frac{\partial p}{\partial T}\right|_V$$

4.2 Microcanonical Ensemble

Fixed energy E gives us a notion of S, T

$$P(n) = \frac{1}{\Omega(E_n)}$$
$$S(E) = k_B \log \Omega(E)$$
$$\frac{1}{T} = \frac{\partial S}{\partial E} \qquad \frac{\partial S}{\partial T} = \frac{C}{T} \qquad p = T\frac{\partial S}{\partial V}$$
$$C = \frac{\partial E}{\partial T} \qquad C_V = \left.\frac{\partial E}{\partial T}\right|_V = T\left.\frac{\partial S}{\partial T}\right|_V \qquad C_p = T\left.\frac{\partial S}{\partial T}\right|_p$$
$$\mathrm{d}E = T\,\mathrm{d}S - p\,\mathrm{d}V$$

4.3 Canonical ensemble

Fixed T gives us an $\langle E \rangle$ ("softly" fixed energy by tuning β)

Boltzmann distrib:

$$P(n) = \frac{e^{-\beta E_n}}{Z} \quad \text{for} \quad Z = \sum_{\text{states}} e^{-\beta E_n}$$

Z multiplicative for independent systems

$$\langle E\rangle = -\partial_\beta \log Z \qquad \Delta E^2 = \partial_\beta^2 \log Z = k_B T^2 C_V \sim \sqrt{N}$$
$$S = -k_B \sum_n P(n) \log P(n) = k_B \partial_T (T \log Z)$$

where the last equality holds for the Boltz dist
reduces to microcanon def if $E = E_\star$ (most likely energy) $= \langle E\rangle$
Free energy

$$F = \langle E\rangle - TS = -\frac{\log Z}{\beta}$$
$$\mathrm{d}F = -S\,\mathrm{d}T - p\,\mathrm{d}V\ (+\mu\,\mathrm{d}N)$$
$$\implies S = -\left.\frac{\partial F}{\partial T}\right|_V \qquad p = -\left.\frac{\partial F}{\partial V}\right|_T$$

with particle number,

$$\mu = -T\left.\frac{\partial S}{\partial N}\right|_{E,V} = \left.\frac{\partial F}{\partial N}\right|_{T,V}$$

4.4 Grand Canonical Ensemble

no longer fix particle number

$$\mathcal{Z} = \sum e^{-\beta(E_n - \mu N_n)} \qquad P(n) = \frac{e^{-\beta E + \beta\mu N}}{\mathcal{Z}}$$

Entropy has the same as in CE, $k_B \partial_T (T \log \mathcal{Z})$. E picks up an extra term:

$$\langle E\rangle - \mu \langle N\rangle = -\partial_\beta \log \mathcal{Z}$$
$$\langle N\rangle = \frac{1}{\beta}\partial_\mu \log \mathcal{Z} \qquad \Delta N^2 = (\frac{1}{\beta}\partial_\beta)^2 \log \mathcal{Z}$$

grand potential

$$\Phi = F - \mu N = E - TS - \mu N = -\frac{1}{\beta}\log \mathcal{Z} = -p(T,\,\mu)V$$
$$\mathrm{d}\Phi = -S\,\mathrm{d}T - p\,\mathrm{d}V - N\,\mathrm{d}\mu$$

we have a pairing of intensive-extensive: TS pV μN, gives E extensive

4.5 Quantum info perspective

have a density matrix instead of probability distribution:

$$\hat{\rho}_C = \frac{1}{Z}\exp\left(-\beta\hat{H}\right) \qquad Z = \mathrm{tr}\left(e^{-\beta\hat{H}}\right)$$
$$\hat{\rho}_{GC} = \frac{1}{\mathcal{Z}}\exp\left(-\beta\hat{H}+\beta\mu\hat{N}\right) \qquad \mathcal{Z} = \mathrm{tr}\left(e^{-\beta\hat{H}+\beta\mu\hat{N}}\right)$$

Grand canon nice in second quant where we have ladder operators for $\hat{N}$

4.6 Classical gases

4.6.1 Monatomic gas

$$Z_1 = \frac{1}{(2\pi\hbar)^3}\int \mathrm{d}^3q\,\mathrm{d}^3p\,e^{-\beta H}$$

in the monatomic case,

$$Z_1 = V\left(\sqrt{\frac{2\pi\hbar^2}{mk_BT}}\right)^{-3} = V/\lambda^3$$

and we get the N-particle gas by $Z_N = Z_1^N = V^N\lambda^{-3N}$

Ideal gas law EoS from $p = -\partial_V F$

equipartition: for each kinetic DoF we have $E \mathrel{+}= \frac{1}{2}k_BT$, (3D = $3N$ DoF)

note: need to account for indistinguishability in the ideal gas partition function:

$$Z_N = \frac{1}{N!}Z_1^N \implies S = Nk_B\left[\log\left(\frac{V}{N\lambda^3}\right)+\frac{5}{3}\right]$$

(sackur-tetrode equation)

adding in a chemical potential, (remember to sum over all N – gives an exp)

$$\mathcal{Z} = \exp\left(e^{\beta\mu}V/\lambda^3\right) \implies \text{(rearranging } N)\ \mu = k_BT\log\left(\lambda^3N/V\right) \qquad \Delta N^2 = N$$

maxwell-boltz distrib (from viewing Z_1 as sum over states of probability):

$$P(v) = 4\pi\left(\frac{m}{2\pi k_BT}\right)^{3/2}v^2e^{-mv^2/2k_BT}$$

gives velocity distribution of a classical gas

4.6.2 Diatomic gas

$$Z_1 = Z_{\text{trans}} Z_{\text{rot}} Z_{\text{vib}}$$

get these new Zs from a phase space integral for the various parts of the hammy

$$Z_{\text{rot}} = \frac{2Ik_BT}{\hbar^2} \implies E_{\text{rot}} = \frac{2}{2}k_BT$$
$$Z_{\text{vib}} = \frac{k_BT}{\hbar\omega} \implies E_{\text{vib}} = \frac{2}{2}k_BT$$

oscillation "freezes out" first, then rotation – limitations of classical equipartition theory (also think about how a deep potential well gives same mechanics as rigid connection, but different degrees of freedom counting. we need the full quantum explanation)

4.6.3 Interacting Gas

virial expansion

$$\beta p = \frac{N}{V} + B_2(T)\frac{N^2}{V^2} + B_3(T)\frac{N^3}{V^3} + \dots$$

define the mayer f function

$$f(r) = e^{-\beta U(r)} - 1$$

allows us to rewrite partition

$$Z_N = \frac{V^N}{N!\lambda^{3N}}\left(1 + \frac{N}{2V}\int \mathrm{d}^3 r\, f(r) + \dots\right)^N$$
$$F = F_{\text{ideal}} - Nk_BT\log\left(1 + \frac{N}{2V}\int f(r)\right)$$

and we find the pressure is

$$p = -\partial_V F = \frac{\rho}{\beta} - \frac{\rho^2}{2\beta}\int f(r)$$

at which point we must pick a U and perform the f integral. typical choice:

$$U(r) = \begin{cases} \infty & r < r_0 \\ -U_0\left(\frac{r_0}{r}\right)^6 & r \geq r_0 \end{cases}$$

which gives

$$\frac{pV}{Nk_BT} = 1 - \frac{N}{V}\left(\frac{a}{k_BT} - b\right) \iff k_BT = \left(p + \frac{N^2}{V^2}a\right)\left(\frac{V}{N} - b\right)^{-1} \implies p = \frac{Nk_BT}{V - bN} - a\frac{N^2}{V^2}$$

at low density and high temperatures for parameters

$$a = \frac{2\pi r_0^3 U_0}{3} \text{ (attractive } p \text{ reduction)} \qquad b = \frac{2\pi r_0^3}{3} \text{ (excluded volume)}$$

higher corrections by cluster expansion

4.7 Quantum gases

DENSITY OF STATES: "if instead of integrating over states, i want to integrate over energies, what do i need as a prefactor?"

$$\sum_n \sim \int \mathrm{d}^3 n = \int \frac{\mathrm{d}^3 x\, \mathrm{d}^3 k}{(2\pi)^3} = \frac{4\pi V}{(2\pi)^3}\int_0^\infty \mathrm{d}k\ k^2 = \frac{V}{2\pi^2}\int \mathrm{d}E \sqrt{\frac{2mE}{\hbar^2}}\frac{m}{\hbar^2} = \int \mathrm{d}E\, g(E)$$

for the usual dispersion relation

$$E = \frac{\hbar^2 k^2}{2m} \implies g(E) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E}$$

or relativistic

$$E = \sqrt{k^2 + m^2} \implies g(E) = \frac{VE}{2\pi^2\hbar^3c^3}\sqrt{E^2 - m^2c^4} \overset{E\gg m}{\approx} \frac{V}{\pi^2\hbar^3c^3}\left(E^2 - \frac{m^2c^4}{2} + \dots\right)$$

4.7.1 photon Gas

photons: idea is to have Z_ω for each frequency, sum over occupation:

$$Z_\omega = \sum_{n=0}^\infty e^{-\beta n\hbar\omega} = \frac{1}{1 - e^{-\beta\hbar\omega}}$$

giving

$$\log Z = \int_0^\infty \mathrm{d}\omega\, g(\omega)\log Z_\omega = -\frac{V}{\pi^2c^3}\int_0^\infty \mathrm{d}\omega\, \omega^2 \log\left(1 - e^{-\beta\hbar\omega}\right)$$

whence we find the Planck distribution of energy,

$$E = -\partial_\beta \log Z = \frac{V\hbar}{\pi^2 c^3} \int_0^\infty \mathrm{d}\omega \, \frac{\omega^3}{e^{\beta\hbar\omega} - 1} = \frac{\pi^2 V (k_B T)^4}{15\hbar^3 c^3}$$

and wien's law, $\omega_{\max} \sim 1/\beta\hbar$. we also get stefan-boltz,

$$\text{energy flux} = \frac{Ec}{4V} = \left(\frac{\pi^2 k_B^4}{60\hbar^3 c^2} \right) T^4$$

free energy gives us pressure, entropy, heat capacity

4.7.2 Bose Gas

$$\mathcal{Z} = \prod_r \frac{1}{1 - e^{-\beta(E_r - \mu)}} \implies \langle n_r \rangle = \frac{1}{e^{\beta(E_r - \mu)} - 1}$$

only makes sense when $\mu < 0$, or fugacity $z = e^{\beta\mu} \in (0, 1)$

doing the usual,

$$N = \int \mathrm{d}E \, \frac{g(E)}{z^{-1} e^{\beta E} - 1} \qquad E = \int \mathrm{d}E \, \frac{E g(E)}{z^{-1} e^{\beta E} - 1}$$

$$pV = -F = -\frac{1}{\beta} \int \mathrm{d}E \, g(E) \log\left(1 - z e^{-\beta E}\right) = \frac{2}{3} E = \frac{V k_B T}{\lambda^3} g_{5/2}(z)$$

where we integrate the log using an IBP: $\mathrm{d}E \, g(E) \sim \mathrm{d}(E^{3/2}) \sim \mathrm{d}(E g(E))$

high-temp (small z) expansion of density:

$$\frac{N}{V} = \frac{z}{\lambda^3} \left(1 + \frac{z}{2\sqrt{2}} + \dots \right)$$

$$\xrightarrow{invert} z = \frac{\lambda^3 N}{V} \left(1 - \frac{1}{2\sqrt{2}} \frac{\lambda^3 N}{V} + \dots \right)$$

gives equation of state

$$pV = N k_B T \left(1 - \frac{\lambda^3 N}{4\sqrt{2} V} + \dots \right)$$

bosons reduce pressure!

4.7.3 BECs

our $\int \mathrm{d}E\,\sqrt{E}$ kills $E = 0$ states when we try to sum over momenta; manually add in

$$N = \frac{V}{\lambda^3} g_{3/2}(z) \to N = \frac{V}{\lambda^3} g_{3/2}(z) + \underbrace{\frac{z}{1-z}}_{\langle n_0 \rangle}$$

(g is a polylog – numerical integration factor. $g_n(1) = \zeta(n)$). Fix parameters st

$$\rho > \lambda^{-3}\zeta(3/2) \geq \rho_{\text{excited}}$$

which lets ρ_{gs} make up for the difference; leads to the above expression for N so long as $\rho \geq \rho_c = \lambda^{-3}\zeta(3/2)$ ("critical density"). below this density, $\mu < 0$ strictly and we have the usual bose gas form. at and above, however, $\mu = 0$ and we get ground state occupancy

GS occupancy has

$$\frac{n_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$

for T_c the temp when $z = 1$. let's see C_V:

$$C_V = \frac{15Vk_B}{4\lambda^3} g_{5/2}(z) - b\left(\frac{T - T_c}{T_c}\right)$$

after a lot of approximations. C_V continuous but its derivative is not – first order pt

4.7.4 Fermi Gas

$$\mathcal{Z} = \prod_r (1 + ze^{-\beta E_r}) \implies n_r = \frac{1}{z^{-1}e^{\beta E} + 1}$$

no restrictions on μ anymore. $g(E)$ carries spin degeneracy $g_s = 2s + 1$

$$g(E) = \frac{g_s V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E}$$

and we have the usual

$$N = \int \mathrm{d}E\,\frac{g(E)}{z^{-1}e^{\beta E} + 1} \qquad E = \int \mathrm{d}E\,\frac{Eg(E)}{z^{-1}e^{\beta E} + 1}$$
$$pV = \frac{1}{\beta}\int \mathrm{d}E\,g(E)\log\left(1 + ze^{-\beta E}\right) = \frac{2}{3}E$$

with the small z EoS

$$pV = Nk_BT\left(1 + \frac{\lambda^3 N}{4\sqrt{2}g_s V} + \dots\right)$$

fermions *increase* the pressure (by the same factor!)

in the $T \to 0$ limit, we have states filled until the fermi energy $E_F = \mu(T = 0)$ — though μ isnt really a function of T, the condition on keeping N fixed allows us to write one in terms of the other (write N as integral up to the surface)

$$E_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{g_s}\frac{N}{V}\right)^{2/3}$$

and we can compute

$$pV = \frac{2}{3}E = \frac{2}{3}\int_0^{E_F} \mathrm{d}E\, Eg(E) = \frac{2}{3}\left(\frac{3}{5}NE_F\right)$$

which is a nonzero "degeneracy" pressure at $T = 0$

in $T \ll T_F$, we can take the integrals to infinity instead of cutting them off. Only states within k_BT of the fermi surface are affected by the temperature, so we can evaluate derivatives of the distribution at E_F; this is the only place it changes.

$$C_V = \left.\frac{\partial E}{\partial T}\right|_{N,V} \sim Tg(E_F) = Nk_B\frac{\pi^2}{2}\frac{T}{T_F}$$

(idea: we have $g(E_F)k_BT$ particles contributing to the physics, each of which has $E \sim k_BT$ – this gives linear heat capacity)

we often combine this linear electronic contribution with the cubic phononic contribution (subsection 4.8) to get the full heat capacity of metals.

to do this low temp expansion rigorously, we sommerfeld expand some polylogs

$$\frac{N}{V} = \frac{g_s}{\lambda^3}f_{3/2}(z) \quad \text{and} \quad \frac{E}{V} = \frac{3}{2}\frac{g_s}{\lambda^3}f_{5/2}(z)$$

the expansion tells us the low-temp expansion in $1/\log(z) = 1/\beta\mu$

$$f_n(z) = \frac{(\log z)^n}{\Gamma(n+1)}\left(1 + \frac{\pi^2}{6}\frac{n(n-1)}{(\log z)^2} + \dots\right)$$

whence we can find

$$\frac{E}{N} = \frac{3E_F}{5}\left(1 + \frac{5\pi^2}{12}\left(\frac{k_BT}{E_F}\right)^2 + \dots\right)$$

and get the heat capacity above.

4.7.5 Diatomic gas

rotation: (recall $2j+1$ degeneracy, sum over all j)

$$E_{\text{rot}} = \frac{\hbar^2}{2I} j(j+1) \implies Z_{\text{rot}} \approx \begin{cases} \frac{2I}{\beta\hbar^2} & T \gg \hbar^2/2Ik_B \\ 1 & T \ll \hbar^2/2Ik_B \end{cases}$$

vibration:

$$E_{\text{vib}} = \hbar\omega(n+1/2) \implies Z_{\text{vib}} = \frac{1}{2\sinh(\beta\hbar\omega/2)} \approx \begin{cases} 1/\beta\hbar\omega & \text{high } T \\ \exp(-\beta\hbar\omega/2) & \text{low } T \end{cases}$$

where the low T gives zero-point energy of QHO and doesnt contribute to C_V

4.8 Debye model of solids

basically just follows from a linear dispersion (and polarization degeneracy)

$$E = \hbar\omega = \hbar k c_s \implies g(\omega) = \frac{3V}{2\pi^2 c_s^3}\omega^2$$

integrals taken up to a cutoff frequency ω_D. To determine the cutoff, consider

$$3N\text{atomic dof} \implies \text{3N phonon dof} = \#\text{one-phonon states} = \int_0^{\omega_D} \mathrm{d}\omega\, g(\omega)$$

which gives

$$\omega_D = \left(\frac{6\pi^2 N}{V}\right)^{1/3} c_s$$

and we can find energy and heat capacity the usual ways.

$$C_V = \begin{cases} Nk_B \frac{12\pi^4}{5}\left(\frac{T}{T_D}\right)^3 & T \ll T_D \\ 3Nk_B & T \gg T_D \end{cases}$$

in low temp limit, integrate to infinity; in high temp limit expand integrand

4.9 electronic gas in B field

4.9.1 pauli paramagnetism

effect from spin coupling to B:

$$E \to E + \underbrace{\frac{|e|\hbar}{2m}}_{\mu_B} Bs$$

we can compute high temp ($z \sim 0$) magnetization

$$M = -\frac{\partial E}{\partial B} = -\mu_B(N_\uparrow - N_\downarrow) \approx \frac{2\mu_B V z}{\lambda^3} \sinh(\beta\mu_B B) \approx \mu_B N \tanh(\beta\mu_B B)$$

and susceptibility

$$\chi = \left.\frac{\partial M}{\partial B}\right|_{B=0} = \frac{N\mu_B^2}{k_B T}$$

at low temps, use expansion of $f_n(z)$ to find

$$M \approx \frac{\mu_B^2 V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{E_F} B \approx \mu_B^2 g(E_F) B$$
$$\chi \approx \mu_B^2 g(E_F) > 0$$

idea: only the $g(E_F)$ electrons on the surface are free to flip

4.9.2 Landau diamagnetism

effect from lorentz force (taking B in the $+z$ direction)

$$H = \frac{1}{2m}(p + eA)^2$$

solving the eigenvalue problem says energy states come in landau levels

$$E = \hbar\omega_c\left(n + \frac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m} \quad \text{for} \quad n \in \mathbb{Z}$$

which have degeneracy

$$\frac{L^2 B}{2\pi\hbar/e} = \frac{\phi}{\phi_0} = \frac{\text{total flux}}{\text{flux quantum}}$$

we proceed to compute the magnetism

$$M = \frac{1}{\beta}\frac{\partial \log \mathcal{Z}}{\partial B} = -\frac{\mu_B^2}{3} g(E_F) B$$

using the partition function

$$\begin{aligned}\log \mathcal{Z} &= \frac{L}{2\pi}\int \mathrm{d}k_z \sum_n \frac{2L^2 B}{\phi_0} \log\left[1 + z\exp\left(-\frac{\beta\hbar^2 k_z^2}{2m} - \beta\hbar\omega_c(n+1/2)\right)\right] \\ &\approx \frac{Vm}{2\pi^2\hbar^2}\left[(\text{const in } B) - \frac{(\hbar\omega_c)^2}{24}\int \mathrm{d}k\, \frac{\beta}{\exp\left[\beta(\hbar^2k^2/2m - \mu)\right] + 1}\right]\end{aligned}$$

this is comparable to pauli but of an opposite sign.

4.10 Phase transitions: van der waals gas

isotherms have that weird wiggle in a $p - v$ diagram below the critical temperature: thus the transition is marked by

$$\frac{\mathrm{d}p}{\mathrm{d}v} = \frac{\mathrm{d}^2 p}{\mathrm{d}v^2} = 0$$

below the critical point, we have weird compressibility and it's broken: we use maxwell's "lol just draw a straight line then" perscription (which comes from setting liquid and gas in chemical equilibrium, $\mu_\ell = \mu_g$ – can also equate GFE per particle)

clausius-clapeyron equation from looking at $p - T$ graph. coexistence region from $p - v$ squeezed into a line (think about traversing an isobar in the $p - v$ diagram and what it means in $p - T$ space). equality of gibbs gives

$$\frac{\mathrm{d}p}{\mathrm{d}T} = \frac{s_g - s_\ell}{v_g - v_\ell} = \frac{L}{T(v_g - v_\ell)}$$

where we've defined the specific latent heat

$$L = T(s_g - s_\ell)$$

this applies to any first-order transition; here we have

$$S = -\frac{\partial F}{\partial T} \quad \text{or} \quad V = \frac{\partial G}{\partial p}$$

as our first-derivative discontinuities

note that S discontinuous $\implies$ $C \sim \partial_T S$ goes to infinity – the temperature doesnt change as we pour heat into the system

We can solve the CC equation with a few assumptions (ideal gas, $v_g \gg v_\ell$, L constant).

$$\frac{\mathrm{d}p}{\mathrm{d}T} = \frac{Lp}{k_B T^2} \implies p = p_0 e^{-L/k_B T}$$

really slick way of getting the critical point: start by rearrangign VdW:

$$p = \frac{Nk_B T}{V - bN} - a\frac{N^2}{V^2} \iff pv^3 - (pb + k_B T)v^2 + av - ab = 0$$

the critical point is defined by $\partial_v p = \partial_v^2 p = 0$, so at the critical temperature, we only have this cubic term:

$$p_c(v - v_c)^3 = 0 = p_c v^3 - (p_c b + k_B T_c)v^2 + av - ab$$

and we can compare term by term in v to get

$$k_B T_c = \frac{8a}{27b} \qquad v_c = 3b \qquad p_c = \frac{a}{27b^2}$$

another handy way of rewriting vdw is in terms of reduced variables; we divide by the critical value, and the equation takes the form

$$\bar{p} = \frac{8}{3}\frac{\bar{T}}{\bar{v} - 1/3} - \frac{3}{\bar{v}^2}$$

which is the path toward the critical exponents

$$v_g - v_\ell \sim (T_c - T)^{1/2} \qquad p - p_c \sim (v - v_c)^3 \qquad \kappa = -\frac{1}{v}\frac{\partial v}{\partial p}\bigg|_T \sim (T - T_c)^{-1}$$

4.11 ising model

$$E = -J\sum_{\langle ij\rangle} s_i s_j - B\sum_i s_i$$

where we're interested in

$$m = \frac{1}{N}\sum \langle s_i \rangle = \frac{1}{N\beta}\frac{\partial \log Z}{\partial B}$$

mean field approximation: write spins in terms of deviation from average and assume that fluctuations are small

$$s_i s_j = \cancel{(s_i - m)(s_j - m)} + m(s_j - m) + m(s_i - m) + m^2$$

so the energy becomes

$$E = \frac{1}{2} JNqm^2 - \underbrace{(Jqm + B)}_{B_{\text{eff}}} \sum s_i$$

and we find, since each spin acts independently,

$$Z = e^{-\frac{1}{2}\beta JNqm^2} 2^N \cosh^N \beta B_{\text{eff}} \implies m = \tanh(\beta B + \beta Jqm)$$

4.11.1 zero magnetic field

when $\beta Jq < 1$ the only solution for m is $m = 0$: there is no average magnetization at high temperatures. if the temperature is low enough, however, we have an unstable solution at $m = 0$ and two stable solutions at $m = \pm m_0$, and in the limit of zero temp $m \to \pm 1$ (all spins aligned). as we vary T, we have a singularity in $\partial_T m$:

second order transition as we vary T

(note: high temperature expansion gets into some stat field theory and RG stuff – possibly important to know?)

4.11.2 nonzero magnetic field

there is no longer a phase transition for a fixed B as T varies: at large temps, magnetization to zero as

$$m \sim \frac{B}{k_B T}$$

and at small temps all spins align with the B field (no choice to make). drawing an $m - T$ graph shows how turning on B separates and smooths out what was a singularity in the $B = 0$ case. however, if we vary B and swap its direction, the magnetization (a first derivative) jumps discontinuously:

first order transition as we vary B from negative to positive and $T < T_C = Jq/k_B$

the critical exponents we get

$$m_0 \sim \pm(T_c - T)^{1/2} \qquad m \sim B^{1/3} \qquad \chi \sim (T - T_c)^{-1}$$

are the same as for VdW

4.12 Polymers

Simplest model: the polymer as a random walk. You get a binomial distribution, which approaches a gaussian in the large N limit. in 1D,

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-x^2/2\sigma^2\right) \xrightarrow{\sigma^2 = Na^2} \frac{1}{\sqrt{2\pi N}a} \exp\left(-x^2/2Na^2\right)$$

for d dimensions each σ_x^2 gets divided by d since the total σ^2 is basically the sums of the individual dimensional walks – in each dimension you only have to walk $1/d$ of the way there[1]

Microcanonical perspective fix X and calculate $F(X)$

$$\mathrm{d}U = 0 = T\,\mathrm{d}S + F\,\mathrm{d}X$$

so if we can find the entropy we can take a derivative to find $F(X)$. The number of states that have a length X is just $N \cdot P(x)$, so entropy comes immediately from the gaussian above.

Canonical perspective fixing F to calculate X

we note that $X = \sum a \cos\theta_i$, so we can just use $E = -FX$

4.13 Brownian motion

idea: large particle of radius b suspended in a fluid. we have a stokes law velocity-dependent damping force and some random force that is time-uncorrelated, ie $\langle F(t)F(t')\rangle = c\,\delta(t-t')$

$$m\partial_t^2 r + 6\pi\eta b\partial_t r = \partial_t^2 r + \alpha\partial_t r = F$$

can reduce the order and get solutions

$$v(t) = A(t)e^{-\alpha t} \implies \dot{A} = e^{\alpha t}F/m$$

where we can now pick out a $v(t)$ by integrating $\dot{A}$ back again

$$v(t) = \frac{1}{m}\int_0^t \mathrm{d}s\, e^{-\alpha(t-s)}F(s)$$

relating our random variable F to a new random variable v. For a fixed T, we can determine the c in $\langle FF\rangle$: we calculate $\langle v^2\rangle$ both according to this description of v and from taking a boltzmann (canonical ensemble) probability distribution.

[1] this is not valid reasoning but it's a way to remember it

5 Assorted mathematical facts to know

if you dont know how to do an integral,
adimensionalize it to pull out all the physics
and just turn it into a number

vector calc thing i always forget:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

a tricky taylor expansion:

$$\frac{1}{|a-r|} = \frac{1}{\sqrt{a^2 - 2a\cdot r + r^2}} = \frac{1}{|a|}\frac{1}{\sqrt{12 - 2a\cdot r/a^2 + r^2/a^2}} \approx \frac{1}{|a|}\left(1 + \frac{r\cdot a}{a^2}\right)$$

common integral:

$$\int_0^\infty \mathrm{d}x\, x^n e^{-x} = n!$$

Stirling's approximation:

$$\log N! \approx N\log N - N$$

Laplace's method/ saddle point integration:

$$\int h(x) e^{Mg(x)} \approx \sqrt{\frac{2\pi}{M|g''(x_0)|}} h(x_0) e^{Mg(x_0)}$$

for M large and x_0 the location of the maximum of g

Spherical coordinates

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

:b:aussian

$$\int \mathrm{d}^n x \exp\left(-\frac{1}{2}\mathbf{x}\cdot\mathbb{A}\cdot\mathbf{x} + \mathbf{v}\cdot\mathbf{x}\right) = \sqrt{\frac{(2\pi)^n}{\det \mathbb{A}}} \exp\left(\frac{1}{2}\mathbf{v}\cdot\mathbb{A}^{-1}\cdot\mathbf{v}\right)$$